

# Global Regularity of 2D almost resistive MHD Equations

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## Abstract

Whether or not the solution to 2D resistive MHD equations is globally smooth remains open. This paper establishes the global regularity of solutions to the 2D almost resistive MHD equations, which require the dissipative operators  $\mathcal{L}$  weaker than any power of the fractional Laplacian. The result is an improvement of the one of Fan et al. (Global Cauchy problem of 2D generalized MHD equations, Monatsh. Math., 175 (2014), pp. 127-131) which ask for  $\alpha > 0, \beta = 1$ .

**Key words:** Almost resistive MHD equations, global regularity.

## 1 Introduction

Consider the Cauchy problem of the two-dimensional generalized magnetohydrodynamic equations:

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + b \cdot \nabla b - \nu \Lambda^{2\alpha} u, \\ b_t + u \cdot \nabla b = b \cdot \nabla u - \kappa \Lambda^{2\beta} b, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x) \end{cases} \quad (1.1)$$

for  $x \in \mathbb{R}^2$  and  $t > 0$ , where  $u = u(x, t)$  is the velocity,  $b = b(x, t)$  the magnetic,  $p = p(x, t)$  the pressure, and  $u_0(x), b_0(x)$  with  $\operatorname{div} u_0(x) = \operatorname{div} b_0(x) = 0$  are the initial

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velocity and magnetic, respectively. Here  $\nu, \kappa, \alpha, \beta \geq 0$  are nonnegative constants and  $\Lambda = \sqrt{-\Delta}$ .

The global regularity of the d-D GMHD (1.1) has attracted a lot of attention and progress has been made in the last few years (see [1-8, 12-20, 22, 23, 25, 26, 28-36]). In 2D case, it follows from [6, 20, 12] that the problem (1.1) has a unique global regular solution if  $\alpha = 0, \beta > 1$  or  $\alpha > 0, \beta = 1$ . In 2D or 3D case, there have been various results on partial regularity, Serrin type regularity criterions for weak solutions, or blow-up criterions for smooth solution to the usual MHD equations, for example [1, 5, 7, 8, 15, 16, 23]. Recently, some important progresses have been made on the global well-posedness for non-resistive MHD equations ( $\kappa = 0, \alpha = 1$ ) near an equilibrium (see [17, 25, 26, 28, 33, 36]). Local existence for 2D non-resistive MHD equations in rough spaces have been obtained in [18, 13, 2, 14]. Some results on global regularity of 2D MHD equations with partial viscosity and resistivity refer to [3, 4]. To the best of our knowledge, whether or not there exists an global regular solution for 2D resistive MHD ( $\nu = 0, \beta = 1$ ) is still an open problem.

In this paper, motivated by [9], we are concerned with the following 2D GMHD

$$\begin{cases} u_t + u \cdot \nabla u + \mathcal{L}u = -\nabla p + b \cdot \nabla b, \\ b_t + u \cdot \nabla b - \Delta b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x). \end{cases} \quad (1.2)$$

where  $\mathcal{L}$  is the dissipative operator with

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^2} \frac{u(x) - u(x-y)}{|y|^{2m}(|y|)} dy. \quad (1.3)$$

Here  $m : [0, \infty) \rightarrow [0, \infty)$  is a smooth, non-decreasing function that behaves like  $\frac{1}{(-\log r)^{1+\varepsilon_1}}$  for sufficiently small  $r$  with  $\varepsilon_1 > 0$  and that grows fast at least at the rate of  $(\log r)^{1+\varepsilon_2}$  for sufficiently large  $r$  with  $\varepsilon_2 > 0$ , satisfying

$$\int_0^1 \frac{m(r)}{r} dr < \infty \quad (1.4)$$

and the doubling condition

$$m(2r) < cm(r) \quad (1.5)$$

for some positive constants  $c$ .

The main result of this paper is stated as follows.

**Theorem 1.1.** Let  $m(r)$  satisfy (1.3)-(1.5) and  $\rho \geq 4$ . Assume that  $u_0, b_0 \in H^\rho(\mathbb{R}^2)$  with  $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ . Then for any  $T > 0$ , the Cauchy problem (1.2) has a unique regular solution

$$(u, b) \in C([0, T]; H^\rho(\mathbb{R}^2)) \text{ and } b \in L^2([0, T]; H^{\rho+2}(\mathbb{R}^2)).$$

The existence and uniqueness are standard we omit their proofs, and only give the a priori estimates.

*Remark 1.1.* Due to

$$\Lambda^{2\alpha}u(x) = c_\alpha P.V. \int_{\mathbb{R}^2} \frac{u(x) - u(x-y)}{|y|^{2+2\alpha}} dy \quad (1.6)$$

for  $\alpha \in (0, 1)$  (see [10]), the dissipative operator  $\mathcal{L}$  defined in Theorem 1.1 is weaker than any power of the fractional Laplacian. Thus we improve the results in [12] for equations (1.1) which require  $\alpha > 0, \beta = 1$ .

*Remark 1.2.* Inspired by the work [9, 21], (1.4) can be replaced by weaker conditions  $\lim_{r \rightarrow 0^+} m(r) = 0$ , then we can obtain the global regularity of solutions to (1.2) with arbitrary weak dissipation  $\mathcal{L}$  (see Remark 3.1).

*Remark 1.3.* In virtue of Remark 2.1 and Section 3, we require only  $u_0, b_0 \in H^\rho(\mathbb{R}^2)$  with  $\rho > 3$ .

*Remark 1.4.* For the 2D GMHD (1.2), it remains an open problem whether there exists a global smooth solution without the dissipative operator  $\mathcal{L}$ .

## 2 Preliminaries

Let us first consider the heat equation

$$\begin{cases} v_t - \Delta v = f, \\ v(x, 0) = v_0(x). \end{cases}$$

As we all know

$$\begin{aligned} v(x, t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta}f(\cdot, s)ds \\ &= h(\cdot, t) * v_0 + \int_0^t h(\cdot, t-s) * f(\cdot, s)ds, \end{aligned} \quad (2.7)$$

where  $h(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$ .

Recalled the following maximal  $L^p(L^q)$  regularity theorem for the heat kernel.

**Lemma 2.1.** ([24]) *Assume  $f \in L^p((0, T), L^q(\mathbb{R}^d))$  ( $1 < p, q < \infty$ ). Let*

$$A : v \mapsto Af(x, t) = \int_0^t e^{(t-s)\Delta} \Delta f(\cdot, s) ds,$$

*then*

$$\|Af\|_{L^p((0, T), L^q(\mathbb{R}^d))} \leq C \|f\|_{L^p((0, T), L^q(\mathbb{R}^d))}$$

*for every  $T \in (0, \infty]$  and some positive constants  $C$  (independent of  $T$ ).*

**Lemma 2.2.** *Let  $u_0, b_0 \in L^2(\mathbb{R}^2)$ , for any  $T > 0$  and  $0 < t < T$ , we have*

$$\begin{aligned} & \|u\|_{L^2}^2(t) + \|b\|_{L^2}^2(t) \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x, \tau) - u(x - y, \tau)|^2}{|y|^2 m(|y|)} dx dy d\tau + \int_0^t \|\nabla b\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \end{aligned}$$

Due to

$$\int_{\mathbb{R}^2} u \mathcal{L} u dx = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|u(x, t) - u(x - y, t)|^2}{|y|^2 m(|y|)} dx dy,$$

we get the Lemma 2.2 easily by the standard  $L^2$ -energy estimates.

Denote  $\omega = \nabla^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2$  the vorticity of the velocity fields and  $j = \nabla^\perp \cdot b = -\partial_2 b_1 + \partial_1 b_2$  the current of the magnetic fields. Applying  $\nabla^\perp \cdot$  on both sides of the equations (1.2), we obtain the following equations for  $\omega$  and  $j$ :

$$\omega_t + u \cdot \nabla \omega + \mathcal{L} \omega = b \cdot \nabla j, \quad (2.8)$$

$$j_t + u \cdot \nabla j - \triangle j = b \cdot \nabla \omega + T(\nabla u, \nabla b), \quad (2.9)$$

where

$$T(\nabla u, \nabla b) = 2\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2 (\partial_1 b_2 + \partial_2 b_1).$$

**Lemma 2.3.** *Let  $u_0, b_0 \in H^1(\mathbb{R}^2)$ . Then for any  $T > 0$  and  $0 < t < T$ , we have*

$$\begin{aligned} & \|\omega\|_{L^2}^2(t) + \|j\|_{L^2}^2(t) \\ & + \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\omega(x, \tau) - \omega(x - y, \tau)|^2}{|y|^2 m(|y|)} dx dy d\tau + \int_0^t \|\nabla j\|_{L^2}^2 d\tau \leq C(T). \end{aligned} \quad (2.10)$$

*Proof.* Multiplying (2.8) by  $\omega$  and (2.9) by  $j$  respectively, integrating and adding together, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\omega(x, t) - \omega(x - y, t)|^2}{|y|^2 m(|y|)} dx dy + \|\nabla j\|_{L^2}^2 \\ & = \int_{\mathbb{R}^2} b \cdot \nabla j \omega dx + \int_{\mathbb{R}^2} b \cdot \nabla \omega j dx + \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j dx \\ & = \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j dx \\ & \leq C \|\nabla u\|_{L^2} \|j\|_{L^4}^2 \\ & \leq C \|\omega\|_{L^2}^2 \|j\|_{L^2}^2 + \frac{1}{2} \|\nabla j\|_{L^2}^2, \end{aligned}$$

where the Gagliardo-Nirenberg inequality has been used in the last inequality.

Thus, we have

$$\begin{aligned} & \frac{d}{dt} (\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\omega(x, t) - \omega(x - y, t)|^2}{|y|^2 m(|y|)} dx dy + \|\nabla j\|_{L^2}^2 \\ & \leq C \|\omega\|_{L^2}^2 \|j\|_{L^2}^2. \end{aligned}$$

By taking advantage of Gronwall inequality and Lemma 2.2, we complete the proof of Lemma 2.3.  $\square$

**Lemma 2.4.** *Let  $u_0, b_0 \in H^2(\mathbb{R}^2)$ . Then for any  $T > 0$  and  $0 < t < T$ , we have*

$$b \in L^\infty((0, T); L^\infty(\mathbb{R}^2)), \nabla b \in L^p((0, T); L^q(\mathbb{R}^2)). \quad (2.11)$$

for any  $p, q \in (2, \infty)$ .

(1.2)<sub>2</sub> can be written as

$$b_t - \Delta b = \sum_{i=1}^2 \partial_i (b_i u - u_i b). \quad (2.12)$$

Due to  $b_i u - u_i b \in L^\infty((0, T); L^p(\mathbb{R}^2))$  and Lemma 2.1, we obtain Lemma 2.4.

**Lemma 2.5.** *Let  $u_0, b_0 \in H^2(\mathbb{R}^2)$ . Then for any  $T > 0$  and  $0 < t < T$ , we have*

$$\omega \in L^\infty((0, T); L^p(\mathbb{R}^2)) \quad (2.13)$$

for any  $p \in (2, \infty)$ .

In virtue of

$$\int_{\mathbb{R}^2} |\omega|^{p-2} \omega(x) \mathcal{L} \omega(x) dx \geq 0$$

for all  $2 \leq p < \infty$  (see [9]), the proof of Lemma 2.5 can be obtained similar to [12].

(2.9) can be encoded by

$$j_t - \Delta j = \sum_{i=1}^2 \partial_i (b_i \omega - u_i j) + T(\nabla u, \nabla b). \quad (2.14)$$

Similar to Lemma 2.4, we have the following lemma.

**Lemma 2.6.** *Let  $u_0, b_0 \in H^2(\mathbb{R}^2)$ . Then for any  $T > 0$  and  $0 < t < T$ , we have*

$$j \in L^\infty((0, T); L^r(\mathbb{R}^2)), \nabla j \in L^q((0, T); L^p(\mathbb{R}^2)) \quad (2.15)$$

for any  $p, q \in (2, \infty)$  and  $r \in (2, \infty]$ .

Exploiting the structure of the (1.2), we can get further estimates.

**Lemma 2.7.** *Let  $p, q \in [2, \infty)$ ,  $r \in [2, \infty]$ . Assume  $u_0, b_0 \in H^4(\mathbb{R}^2)$ , then for any  $T > 0$ , we have*

$$\nabla j \in L^\infty((0, T); L^p(\mathbb{R}^2)), \Delta b + b \cdot \nabla u \in L^\infty((0, T); L^r(\mathbb{R}^2)), \quad (2.16)$$

$$\nabla(\Delta b + b \cdot \nabla u) \in L^q((0, T); L^p(\mathbb{R}^2)). \quad (2.17)$$

*Remark 2.1.* In fact, the estimates of (2.16) need only  $u_0 \in H^{2+\epsilon_3}(\mathbb{R}^2)$ ,  $b_0 \in H^{2+\epsilon_3}(\mathbb{R}^2)$  with  $\epsilon_3 > 0$ .

*Remark 2.2.* Concerning the 2D resistive MHD, we still obtain the estimates (2.16) and (2.17).

For a 2D Euler equation with nonlocal forces

$$\omega_t + u \cdot \nabla \omega = -\partial_y u_1 = \mathcal{R}_{22} \omega,$$

where  $\mathcal{R}_{ij} \omega$  denotes the Riesz transform  $\partial_{ij} \Lambda^{-2} \omega$ . Elgindi and Masmoudi [11] prove that it is mildly ill-posed in  $L^\infty$ .

Similar to (3.25), we have

$$\begin{aligned} \omega_t + u \cdot \nabla \omega &= b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1) - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1 \\ &= f + b_1 \sum_{i=1}^2 b_i \mathcal{R}_{i2} \omega - b_2 \sum_{i=1}^2 b_i \mathcal{R}_{i1} \omega, \end{aligned} \quad (2.18)$$

where  $f = b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1) \in L^\infty(0, T; L^\infty(\mathbb{R}^2))$ . So the results in [11] suggest that it may be mildly ill-posed in  $L^\infty$  in the case of 2D resistive MHD.

*Proof.* Applying  $b \cdot \nabla$  and  $\Delta$  to (1.2)<sub>1</sub> and (1.2)<sub>2</sub> respectively, and multiplying (1.2)<sub>2</sub> by  $\nabla u$ , then adding the resulting equations together we obtain

$$\begin{aligned} &(\Delta b + b \cdot \nabla u)_t - \Delta(\Delta b + b \cdot \nabla u) \\ &= -b \cdot \nabla(u \cdot \nabla u) + b \cdot \nabla(b \cdot \nabla b) - (u \cdot \nabla b) \cdot \nabla u + (b \cdot \nabla u) \cdot \nabla u \\ &\quad + \Delta b \cdot \nabla u - b \cdot \nabla(\nabla p) - \Delta(u \cdot \nabla b) - b \cdot \nabla \mathcal{L}u. \end{aligned} \quad (2.19)$$

Firstly, we give the following estimates

$$\Delta b + b \cdot \nabla u \in L^\infty((0, T); L^2(\mathbb{R}^2)) \bigcap L^2((0, T); H^1(\mathbb{R}^2)).$$

Multiplying (2.19) by  $\Delta b + b \cdot \nabla u$  and integrating on  $\mathbb{R}^2$ , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\Delta b + b \cdot \nabla u\|_{L^2}^2 + \|\nabla(\Delta b + b \cdot \nabla u)\|_{L^2}^2 \\ &= - \int_{\mathbb{R}^2} b \cdot \nabla(u \cdot \nabla u)(\Delta b + b \cdot \nabla u) dx + \int_{\mathbb{R}^2} b \cdot \nabla(b \cdot \nabla b)(\Delta b + b \cdot \nabla u) dx \\ &\quad - \int_{\mathbb{R}^2} (u \cdot \nabla b) \cdot \nabla u(\Delta b + b \cdot \nabla u) dx + \int_{\mathbb{R}^2} (b \cdot \nabla u) \cdot \nabla u(\Delta b + b \cdot \nabla u) dx \\ &\quad + \int_{\mathbb{R}^2} \Delta b \cdot \nabla u(\Delta b + b \cdot \nabla u) dx - \int_{\mathbb{R}^2} b \cdot \nabla(\nabla p)(\Delta b + b \cdot \nabla u) dx \\ &\quad - \int_{\mathbb{R}^2} \Delta(u \cdot \nabla b)(\Delta b + b \cdot \nabla u) dx - \int_{\mathbb{R}^2} b \cdot \nabla \mathcal{L}u(\Delta b + b \cdot \nabla u) dx \\ &= RHS. \end{aligned}$$

Thanks to

$$\begin{aligned}
& \|b_i \mathcal{L}u\|_{L^2} \\
&= \left( \int_{\mathbb{R}^2} \left| b_i(x) P.V. \int_{\mathbb{R}^2} \frac{u(x) - u(x-y)}{|y|^2 m(|y|)} dy \right|^2 dx \right)^{\frac{1}{2}} \\
&= \left( \int_{\mathbb{R}^2} \left| b_i(x) P.V. \int_{|y| \leq 1} \frac{u(x) - u(x-y)}{|y|^2 m(|y|)} dy + b_i(x) P.V. \int_{|y| \geq 1} \frac{u(x) - u(x-y)}{|y|^2 m(|y|)} dy \right|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \left( \int_{\mathbb{R}^2} \left| b_i(x) \int_{|y| \leq 1} \int_0^1 \frac{|(\nabla u)(x - (1-t)y)|}{|y| m(|y|)} dt dy \right|^2 dx \right)^{\frac{1}{2}} \\
&\quad + C \|b\|_{L^2} \|u\|_{L^\infty} \int_{|y| \geq 1} \frac{1}{|y|^2 m(|y|)} dy \\
&\leq C (\|b\|_{L^\infty} \|\nabla u\|_{L^2} + \|b\|_{L^2} \|u\|_{L^\infty}) \tag{2.20}
\end{aligned}$$

and Lemmas 4-6, the right hand side above can be simply estimated as follows

$$\begin{aligned}
RHS &\leq (\|b\|_{L^6} \|u\|_{L^6} \|\nabla u\|_{L^6} + \|b\|_{L^6}^2 \|\nabla b\|_{L^6}) \|\nabla(\Delta b + b \cdot \nabla u)\|_{L^2} \\
&\quad + (\|u\|_{L^6} \|\nabla b\|_{L^6} + \|b\|_{L^6} \|\nabla u\|_{L^6}) \|\nabla u\|_{L^6} \|\Delta b + b \cdot \nabla u\|_{L^2} \\
&\quad + \|\Delta b\|_{L^4} \|\nabla u\|_{L^4} \|\Delta b + b \cdot \nabla u\|_{L^2} + \|b\|_{L^4} \|\nabla p\|_{L^4} \|\nabla(\Delta b + b \cdot \nabla u)\|_{L^2} \\
&\quad + (\|\nabla u\|_{L^4} \|\nabla b\|_{L^4} + \|u\|_{L^4} \|\nabla^2 b\|_{L^4}) \|\nabla(\Delta b + b \cdot \nabla u)\|_{L^2} \\
&\quad + C(\|b\|_{L^\infty} \|\nabla u\|_{L^2} + \|b\|_{L^2} \|u\|_{L^\infty}) \|\nabla(\Delta b + b \cdot \nabla u)\|_{L^2} \\
&\leq c(t) (\|\Delta b + b \cdot \nabla u\|_{L^2} + C(T)) + \frac{1}{2} \|\nabla(\Delta b + b \cdot \nabla u)\|_{L^2}
\end{aligned}$$

where  $c(t) \in L^p(0, T)$  ( $2 \leq p < \infty$ ). Taking advantage of Gronwall inequality, we get the result.

Secondly, we prove the following estimates

$$\Delta b + b \cdot \nabla u \in L^\infty((0, T); L^p(\mathbb{R}^2)) \quad (2 < p < \infty). \tag{2.21}$$

Thus,  $\Delta b \in L^\infty((0, T); L^p(\mathbb{R}^2))$  ( $2 < p < \infty$ ).

(2.19) can be written as

$$(\Delta b + b \cdot \nabla u)(x, t) \triangleq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9, \tag{2.22}$$

where

$$\begin{aligned}
I_1 &= h(\cdot, t) * (\Delta b_0 + b_0 \cdot \nabla u_0), \quad I_2 = - \int_0^t h(\cdot, t-s) * (b \cdot \nabla(u \cdot \nabla u))(\cdot, s) ds, \\
I_3 &= \int_0^t h(\cdot, t-s) * (b \cdot \nabla(b \cdot \nabla b))(\cdot, s) ds, \quad I_4 = - \int_0^t h(\cdot, t-s) * ((u \cdot \nabla b) \cdot \nabla u)(\cdot, s) ds, \\
I_5 &= \int_0^t h(\cdot, t-s) * ((b \cdot \nabla u) \cdot \nabla u)(\cdot, s) ds, \quad I_6 = \int_0^t h(\cdot, t-s) * (\Delta b \cdot \nabla u)(\cdot, s) ds, \\
I_7 &= - \int_0^t h(\cdot, t-s) * (b \cdot \nabla(\nabla p))(\cdot, s) ds, \quad I_8 = - \int_0^t h(\cdot, t-s) * (\Delta(u \cdot \nabla b))(\cdot, s) ds, \\
I_9 &= - \int_0^t h(\cdot, t-s) * (b \cdot \nabla \mathcal{L}u)(\cdot, s) ds.
\end{aligned}$$

Then

$$\begin{aligned}
\|I_1\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} &= \|h(\cdot, t) * (\Delta b_0 + b_0 \cdot \nabla u_0)\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|h\|_{L^\infty((0,T);L^1(\mathbb{R}^2))} \|\Delta b_0 + b_0 \cdot \nabla u_0\|_{L^p(\mathbb{R}^2)} \\
&\leq C \|\nabla^2 b_0\|_{L^p(\mathbb{R}^2)} + C \|b_0\|_{L^{2p}(\mathbb{R}^2)} \|\nabla u_0\|_{L^{2p}(\mathbb{R}^2)} \\
&\leq C(\|u_0\|_{H^p(\mathbb{R}^2)} + \|b_0\|_{H^p(\mathbb{R}^2)}). \tag{2.23}
\end{aligned}$$

$$\begin{aligned}
\|I_2\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} &= \left\| \int_0^t h(\cdot, t-s) * (b \cdot \nabla(u \cdot \nabla u))(\cdot, s) ds \right\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|\nabla h\|_{L^1((0,T);L^1(\mathbb{R}^2))} \|bu \nabla u\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|b\|_{L^\infty((0,T);L^\infty(\mathbb{R}^2))} \|u\|_{L^\infty((0,T);L^\infty(\mathbb{R}^2))} \|\nabla u\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C(T).
\end{aligned}$$

Arguing similarly to above, it can be derived  $\|I_3\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ ,  $\|I_7\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ . Using an argument deriving the estimate (2.20), we have  $\|b_i \mathcal{L}u\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ , so  $\|I_9\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ .

For  $I_4$ , we obtain

$$\begin{aligned}
\|I_4\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} &= \left\| \int_0^t h(\cdot, t-s) * ((u \cdot \nabla b) \cdot \nabla u)(\cdot, s) ds \right\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|h\|_{L^1((0,T);L^1(\mathbb{R}^2))} \|(u \cdot \nabla b) \cdot \nabla u\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|u\|_{L^\infty((0,T);L^{3p}(\mathbb{R}^2))} \|\nabla b\|_{L^\infty((0,T);L^{3p}(\mathbb{R}^2))} \|\nabla u\|_{L^\infty((0,T);L^{3p}(\mathbb{R}^2))} \\
&\leq C(T).
\end{aligned}$$

Similarly,  $\|I_5\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ .

Choosing  $2 < q < \infty$ , one has

$$\begin{aligned}
\|I_6\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} &= \left\| \int_0^t h(\cdot, t-s) * (\Delta b \cdot \nabla u)(\cdot, s) ds \right\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|h\|_{L^{q'}((0,T);L^1(\mathbb{R}^2))} \|\Delta b \cdot \nabla u\|_{L^q((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|\Delta b\|_{L^{2q}((0,T);L^{2p}(\mathbb{R}^2))} \|\nabla u\|_{L^{2q}((0,T);L^{2p}(\mathbb{R}^2))} \\
&\leq C(T),
\end{aligned}$$

$$\begin{aligned}
\|I_8\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} &= \left\| \int_0^t h(\cdot, t-s) * (\Delta(u \cdot \nabla b))(\cdot, s) ds \right\|_{L^\infty((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|\nabla h\|_{L^{q'}((0,T);L^1(\mathbb{R}^2))} \|\nabla(u \cdot \nabla b)\|_{L^q((0,T);L^p(\mathbb{R}^2))} \\
&\leq C \|\nabla u\|_{L^{2q}((0,T);L^{2p}(\mathbb{R}^2))} \|\nabla b\|_{L^{2q}((0,T);L^{2p}(\mathbb{R}^2))} \\
&\quad + C \|u\|_{L^{2q}((0,T);L^{2p}(\mathbb{R}^2))} \|\nabla^2 b\|_{L^{2q}((0,T);L^{2p}(\mathbb{R}^2))} \\
&\leq C(T),
\end{aligned}$$



where  $q$  and  $q'$  satisfy  $\frac{1}{q} + \frac{1}{q'} = 1$  and  $q' < 2$ . So we arrive at (2.21).

Thirdly, we prove

$$\Delta b + b \cdot \nabla u \in L^\infty((0, T); L^\infty(\mathbb{R}^2)). \quad (2.24)$$

For  $I_1$ , similar to (2.23), we have

$$\|I_1\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(\|u_0\|_{H^\rho(\mathbb{R}^2)} + \|b_0\|_{H^\rho(\mathbb{R}^2)}).$$

Let  $2 < p_1 < \infty$  and  $\frac{1}{p_1} + \frac{1}{p'_1} = 1$ .

$$\begin{aligned} \|I_2\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} &= \left\| \int_0^t h(\cdot, t-s) * (b \cdot \nabla(u \cdot \nabla u))(\cdot, s) ds \right\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \\ &\leq C \|\nabla h\|_{L^1((0, T); L^{p'_1}(\mathbb{R}^2))} \|bu \nabla u\|_{L^\infty((0, T); L^{p_1}(\mathbb{R}^2))} \\ &\leq C \|b\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \|u\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \|\nabla u\|_{L^\infty((0, T); L^{p_1}(\mathbb{R}^2))} \\ &\leq C(T). \end{aligned}$$

Similarly,  $\|I_3\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(T)$ ,  $\|I_7\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(T)$ ,  $\|I_8\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(T)$ ,  $\|I_9\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(T)$ .

For  $I_4$ , we have

$$\begin{aligned} \|I_4\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} &= \left\| \int_0^t h(\cdot, t-s) * ((u \cdot \nabla b) \cdot \nabla u)(\cdot, s) ds \right\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \\ &\leq C \|h\|_{L^1((0, T); L^{p'_1}(\mathbb{R}^2))} \|(u \cdot \nabla b) \cdot \nabla u\|_{L^\infty((0, T); L^{p_1}(\mathbb{R}^2))} \\ &\leq C \|u\|_{L^\infty((0, T); L^{3p_1}(\mathbb{R}^2))} \|\nabla b\|_{L^\infty((0, T); L^{3p_1}(\mathbb{R}^2))} \|\nabla u\|_{L^\infty((0, T); L^{3p_1}(\mathbb{R}^2))} \\ &\leq C(T), \end{aligned}$$

Similarly,  $\|I_5\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(T)$ ,  $\|I_6\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \leq C(T)$ . And (2.24) is proved.

Finally, we prove (2.17).

For  $\nabla I_1$ , we have

$$\begin{aligned} \|\nabla I_1\|_{L^q((0, T); L^p(\mathbb{R}^2))} &= \|\nabla(h(\cdot, t) * (\Delta b_0 + b_0 \cdot \nabla u_0))\|_{L^q((0, T); L^p(\mathbb{R}^2))} \\ &\leq C \|h\|_{L^q((0, T); L^1(\mathbb{R}^2))} \|\nabla(\Delta b_0 + b_0 \cdot \nabla u_0)\|_{L^p(\mathbb{R}^2)} \\ &\leq C \|\nabla^3 b_0\|_{L^p(\mathbb{R}^2)} + C \|\nabla b_0\|_{L^{2p}(\mathbb{R}^2)} \|\nabla u_0\|_{L^{2p}(\mathbb{R}^2)} \\ &\quad + C \|b_0\|_{L^{2p}(\mathbb{R}^2)} \|\nabla^2 u_0\|_{L^{2p}(\mathbb{R}^2)} \leq C(T). \end{aligned}$$

Thanks to Lemma 2.1, we obtain

$$\begin{aligned} \|\nabla I_2\|_{L^q((0, T); L^p(\mathbb{R}^2))} &= \left\| \nabla \int_0^t h(\cdot, t-s) * (b \cdot \nabla(u \cdot \nabla u))(\cdot, s) ds \right\|_{L^q((0, T); L^p(\mathbb{R}^2))} \\ &\leq C \|bu \nabla u\|_{L^q((0, T); L^p(\mathbb{R}^2))} \\ &\leq C \|b\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \|u\|_{L^\infty((0, T); L^\infty(\mathbb{R}^2))} \|\nabla u\|_{L^q((0, T); L^p(\mathbb{R}^2))} \\ &\leq C(T). \end{aligned}$$

Similarly,  $\|\nabla I_3\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ ,  $\|\nabla I_7\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ ,  
 $\|\nabla I_8\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ ,  $\|\nabla I_9\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ .

For  $\nabla I_4$ , we get

$$\begin{aligned} \|\nabla I_4\|_{L^q((0,T);L^p(\mathbb{R}^2))} &= \left\| \nabla \int_0^t h(\cdot, t-s) * ((u \cdot \nabla b) \cdot \nabla u)(\cdot, s) ds \right\|_{L^q((0,T);L^p(\mathbb{R}^2))} \\ &\leq C \|\nabla h\|_{L^1((0,T);L^1(\mathbb{R}^2))} \| (u \cdot \nabla b) \cdot \nabla u \|_{L^q((0,T);L^p(\mathbb{R}^2))} \\ &\leq C \|u\|_{L^{3q}((0,T);L^{3p}(\mathbb{R}^2))} \|\nabla b\|_{L^{3q}((0,T);L^{3p}(\mathbb{R}^2))} \|\nabla u\|_{L^{3q}((0,T);L^{3p}(\mathbb{R}^2))} \\ &\leq C(T). \end{aligned}$$

Similarly,  $\|\nabla I_4\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ ,  $\|\nabla I_5\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ ,  
 $\|\nabla I_6\|_{L^q((0,T);L^p(\mathbb{R}^2))} \leq C(T)$ .

Therefore, we obtain (2.17) and finish the proof of lemma 2.7.  $\square$

### 3 The Proof of Theorem 1.1

Due to

$$\partial_1 j = \Delta b_2, \quad \partial_2 j = -\Delta b_1,$$

(2.8) can be changed into

$$\begin{aligned} \omega_t + u \cdot \nabla \omega + \mathcal{L} \omega &= b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1) - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1 \\ &= f - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1, \end{aligned} \quad (3.25)$$

where  $f = b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1)$ .

Multiplying (3.25) by  $\omega(x, t)$ , we obtain

$$\frac{1}{2}(\partial_t + u \cdot \nabla) |\omega(x, t)|^2 + \omega(x, t) \mathcal{L} \omega(x, t) = (f - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1)(x, t) \omega(x, t).$$

Using the pointwise identity

$$\omega(x, t) \mathcal{L} \omega(x, t) = \frac{1}{2} \mathcal{L}(|\omega(x, t)|^2) + \frac{D(x, t)}{2}$$

(see [9]), where

$$D(x, t) = P.V. \int_{\mathbb{R}^2} \frac{(\omega(x, t) - \omega(x - y, t))^2}{|y|^2 m(|y|)} dy,$$

we get

$$\frac{1}{2}(\partial_t + u \cdot \nabla + \mathcal{L}) |\omega(x, t)|^2 + \frac{D(x, t)}{2} = (f - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1)(x, t) \omega(x, t). \quad (3.26)$$

Choosing a non-negative radial smooth cut-off function  $\chi_1(x)$  supported in  $|x| \leq 1$ , identically equal to 1 for  $|x| \leq \frac{1}{2}$  and  $|\nabla \chi_1(x)| \leq C$ . Let  $\chi_2(x) = 1 - \chi_1(x)$ .

By Biot-Savart law [27],

$$u(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(-\frac{y_2}{|y|^2}, \frac{y_1}{|y|^2}\right) \omega(x - y, t) dy, \quad (3.27)$$

so

$$\begin{aligned} & |(b_1 b \cdot \nabla u_2 \omega)(x, t)| \\ &= \left| b_1(x, t) \omega(x, t) b(x, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_1}{|y|^2} \nabla_y \omega(x - y, t) dy \right| \\ &\leq \left| b_1(x, t) \omega(x, t) b(x, t) \cdot \frac{1}{2\pi} \int_{|y| \leq 1} \frac{y_1}{|y|^2} \nabla_y (\omega(x, t) - \omega(x - y, t)) \chi_1(y) dy \right| \\ &+ \left| b_1(x, t) \omega(x, t) b(x, t) \cdot \frac{1}{2\pi} \int_{|y| \geq \frac{1}{2}} \frac{y_1}{|y|^2} \nabla_y \omega(x - y, t) \chi_2(y) dy \right| \\ &\leq c_1 \|b\|_{L^\infty}^2 |\omega(x, t)| \int_{|y| \leq 1} \frac{1}{|y|^2} |\omega(x, t) - \omega(x - y, t)| dy \\ &+ c_2 \|b\|_{L^\infty}^2 |\omega(x, t)| \int_{|y| \geq \frac{1}{2}} \frac{1}{|y|^2} |\omega(x - y, t)| dy \\ &\leq c_1 \|b\|_{L^\infty}^2 |\omega(x, t)| \int_{|y| \leq 1} \frac{|\omega(x, t) - \omega(x - y, t)| \sqrt{m(|y|)}}{|y| \sqrt{m(|y|)}} \frac{\sqrt{m(|y|)}}{|y|} dy + c_3 \|b\|_{L^\infty}^2 \|\omega\|_{L^2} |\omega(x, t)| \\ &\leq c_4 \|b\|_{L^\infty}^2 |\omega(x, t)| \sqrt{D(x, t)} \left( \int_0^1 \frac{m(r)}{r} dr \right)^{\frac{1}{2}} + c_3 \|b\|_{L^\infty}^2 \|\omega\|_{L^2} |\omega(x, t)| \\ &\leq \frac{D(x, t)}{8} + c_5 \|b\|_{L^\infty}^4 |\omega(x, t)|^2 + c_3 \|b\|_{L^\infty}^2 \|\omega\|_{L^2} |\omega(x, t)|. \end{aligned} \quad (3.28)$$

Similarly,

$$\begin{aligned} & |(b_2 b \cdot \nabla u_1 \omega)(x, t)| \\ &\leq \frac{D(x, t)}{8} + c_5 \|b\|_{L^\infty}^4 |\omega(x, t)|^2 + c_3 \|b\|_{L^\infty}^2 \|\omega\|_{L^2} |\omega(x, t)|. \end{aligned}$$

Thus, thanks to Lemma 2.7, (3.26) and (3.28) give

$$\begin{aligned} & \frac{1}{2} (\partial_t + u \cdot \nabla + \mathcal{L}) |\omega(x, t)|^2 + \frac{D(x, t)}{4} \\ &\leq 2c_5 \|b\|_{L^\infty}^4 |\omega(x, t)|^2 + (2c_3 \|b\|_{L^\infty}^2 \|\omega\|_{L^2} + \|f\|_{L^\infty}) |\omega(x, t)| \\ &\leq 2c_5 \|b\|_{L^\infty}^4 |\omega(x, t)|^2 + (2c_3 \|b\|_{L^\infty}^2 \|\omega\|_{L^2} + 2 \|b\|_{L^\infty} \|\Delta b + b \cdot \nabla u\|_{L^\infty}) |\omega(x, t)| \\ &\leq C_1(T) (|\omega(x, t)| + |\omega(x, t)|^2). \end{aligned} \quad (3.29)$$

Since

$$D(x, t) \geq \frac{c_6}{m(1)} |\omega(x, t)|^2 \log \frac{1}{\delta} - c_7 |\omega(x, t)| \|\omega\|_{L^2} \frac{1}{\delta m(\delta)}$$

(see (5.18) in [9]), where  $\delta < 1$ . We pick  $\delta = \delta(m, T) \in (0, 1)$  to be such that

$$\frac{c_6}{8m(1)} \log \frac{1}{\delta} > C_1(T).$$

Hence,

$$\begin{aligned}
& \frac{1}{2}(\partial_t + u \cdot \nabla + \mathcal{L}) |\omega(x, t)|^2 + C_2(T) |\omega(x, t)|^2 \\
& \leq (C_1(T) + C_3(T) \|\omega\|_{L^2}) |\omega(x, t)| \\
& \leq C_4(T) |\omega(x, t)|.
\end{aligned} \tag{3.30}$$

Let  $\varphi(r)$  be a non-decreasing positive convex smooth function which is identically 0 on  $0 \leq r \leq \max\{\|\omega_0\|_{L^\infty}^2, (\frac{C_4(T)}{C_1(T)})^2\}$ . Multiplying (3.30) by  $\varphi'(|\omega(x, t)|^2)$  gives

$$(\partial_t + u \cdot \nabla + \mathcal{L})\varphi(|\omega(x, t)|^2) \leq 0 \tag{3.31}$$

for all  $x$  and all  $t \in [0, T)$ . Thanks to

$$\int_{\mathbb{R}^2} |\omega(x)|^{p-2} \omega(x) \mathcal{L} \omega(x) dx \geq 0,$$

for all  $2 \leq p < \infty$  (see [9]). Hence from (3.31), we obtain

$$\|\varphi(|\omega(x, t)|^2)\|_{L^\infty} \leq \|\varphi(|\omega_0|^2)\|_{L^\infty} = 0.$$

This gives that  $\|\omega(\cdot, t)\|_{L^\infty} \leq \max\{\|\omega_0\|_{L^\infty}, \frac{C_4(T)}{C_1(T)}\}$  for all  $t \in [0, T)$ .

By taking advantage of the BKM type criterion for global regularity (see [1]), we finish the proof of Theorem 1.1.

*Remark 3.1.* Similar to Remark 5.4 in [9], if (1.4) is replaced by  $\lim_{r \rightarrow 0^+} m(r) = 0$ , we can still obtain the global regularity of (1.2). We only give  $\omega \in L^\infty([0, T]; L^\infty(\mathbb{R}^2))$ . A sketch of the proof is as follows. We can assume that  $\sup_{x \in \mathbb{R}^2} \omega(x, t)$  is obtained at  $\bar{x}(t)$  for  $t \in [0, T)$ , if not, we only need to consider (3.25) multiplied by a smooth cut-off function. Then, at  $\bar{x}$  the convection term in (3.25) vanishes and we have

$$\partial_t \omega(\bar{x}, t) + \mathcal{L} \omega(\bar{x}, t) = (f - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1)(\bar{x}, t).$$

Choosing a non-negative radial smooth cut-off function  $\chi_3(x)$  supported in  $|x| \leq \eta$  ( $\eta > 0$ ), identically equal to 1 for  $|x| \leq \frac{1}{2}\eta$  and  $|\nabla \chi_3(x)| \leq \frac{C}{\eta}$ . Let  $\chi_4(x) = 1 - \chi_3(x)$ . Then by (3.27), we have

$$\begin{aligned}
& |(b_1 b \cdot \nabla u_2)(\bar{x}, t)| \\
& = \left| b_1(\bar{x}, t) b(\bar{x}, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_1}{|y|^2} \nabla_y \omega(\bar{x} - y, t) dy \right| \\
& \leq \left| b_1(\bar{x}, t) b(\bar{x}, t) \cdot \frac{1}{2\pi} \int_{|y| \leq \eta} \frac{y_1}{|y|^2} \nabla_y (\omega(\bar{x}, t) - \omega(\bar{x} - y, t)) \chi_3(y) dy \right| \\
& + \left| b_1(\bar{x}, t) b(\bar{x}, t) \cdot \frac{1}{2\pi} \int_{|y| \geq \frac{1}{2}\eta} \frac{y_1}{|y|^2} \nabla_y \omega(\bar{x} - y, t) \chi_4(y) dy \right| \\
& \leq C \|b\|_{L^\infty}^2 \int_{|y| \leq \eta} \frac{1}{|y|^2} |\omega(\bar{x}, t) - \omega(\bar{x} - y, t)| dy \\
& + C \|b\|_{L^\infty}^2 \int_{|y| \geq \frac{1}{2}\eta} \frac{1}{|y|^2} |\omega(\bar{x} - y, t)| dy.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& |(b_1 b \cdot \nabla u_2)(\bar{x}, t)| \\
& \leq C \|b\|_{L^\infty}^2 \int_{|y| \leq \eta} \frac{1}{|y|^2} |\omega(\bar{x}, t) - \omega(\bar{x} - y, t)| dy \\
& + C \|b\|_{L^\infty}^2 \int_{|y| \geq \frac{1}{2}\eta} \frac{1}{|y|^2} |\omega(\bar{x} - y, t)| dy.
\end{aligned}$$

Hence,

$$\begin{aligned}
\partial_t \omega(\bar{x}, t) & \leq \int_{\mathbb{R}^2} \frac{\omega(\bar{x} - y, t) - \omega(\bar{x}, t)}{|y|^2 m(|y|)} dy + C \|b\|_{L^\infty}^2 \int_{|y| \leq \eta} \frac{1}{|y|^2} |\omega(\bar{x}, t) - \omega(\bar{x} - y, t)| dy \\
& + C \|b\|_{L^\infty}^2 \int_{|y| \geq \frac{1}{2}\eta} \frac{1}{|y|^2} |\omega(\bar{x} - y, t)| dy + f(\bar{x}, t) \\
& \leq \int_{|y| \leq \eta} \frac{\omega(\bar{x} - y, t) - \omega(\bar{x}, t)}{|y|^2} \left( \frac{1}{m(|y|)} - C \|b\|_{L^\infty}^2 \right) dy \\
& - \omega(\bar{x}, t) \int_{|y| \geq \eta} \frac{1}{|y|^2 m(|y|)} dy + C \frac{1}{\eta} (\|b\|_{L^\infty}^2 + \frac{1}{m(\eta)}) \|\omega\|_{L^2} + \|f\|_{L^\infty}.
\end{aligned}$$

Thanks to Lemma 2.4, we can choose  $\eta$  (dependent on  $T$ ) small so that  $\frac{1}{m(|y|)} - C \|b\|_{L^\infty}^2 > 0$ . Due to Lemma 2.7, we obtain

$$\partial_t \omega(\bar{x}, t) \leq C_5(T) - C_6(T) \omega(\bar{x}, t).$$

Therefore  $\omega(\bar{x}, t) \leq C(T)$  for  $t \in [0, T)$ . A similar argument can be applied to the minimum and we obtain  $\|\omega(\cdot, t)\|_{L^\infty} \leq C(T)$  for all  $t \in [0, T)$ .

**Acknowledgement** The research of B Yuan was partially supported by the National Natural Science Foundation of China (No. 11471103).

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